Simultaneous Quasidiagonalization of Complex Matrices

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ABSTRACT

Let $\mathcal{Q}$ be the complex algebra generated by a pair of $n\times n$ Hermitian matrices A, B. A recent result of Watters states that A,B are simultaneously unitarily quasidiagonalizable [i.e., A and B are simultaneously unitarily similar to direct sums $C_1\oplus\cdots\oplus C_t$, $D_1\oplus\cdots\oplus D_t$ for some t, where C_i,D_i are $k_i\times k_i$ and $k_i\leqslant 2$ $(1\leqslant i\leqslant t)$] if and only if $[p(A,B),A]^2$ and $[p(A,B),B]^2$ belong to the center of $\mathcal{Q}$ for all polynomials $p(x,y)$ in the noncommuting variables x,y. In this paper, we obtain a *finite set* of conditions which works. In particular we show that if A,B are positive semidefinite, then A,B are simultaneously quasidiagonalizable if (and only if) $[A,B]^2$, $[A^2,B]^2$ and $[A,B^2]^2$ commute with A,B.

In [5] J. F. Watters obtained necessary and sufficient conditions for a set $\mathcal{S}$ of normal matrices to be simultaneously unitarily quasidiagonalizable [i.e., there exists a unitary matrix U such that for every $S\in\mathcal{S}$, $U^{-1}SU=S_1\oplus\cdots\oplus S_r$, where S_i is 2×2 $(1\leqslant i\leqslant r-1)$, and S_r is 1×1 or 2×2, depending on the parity of n]. In this paper we are concerned with obtaining a *finite set* of conditions which works when $\mathcal{S}$ consists of a pair of Hermitian matrices.

Before stating our first lemma, we recall several results on the representation theory of finite-dimensional algebras. A good reference for this material is Huppert's book [1, Chapter V, Secs 1–4].

Let R be a ring, and let $M\neq 0$ be a right R-module. Then M is called *indecomposable* if M cannot be written as the direct sum of two nonzero R-submodules. We state

I (Krull-Remak-Schmidt theorem [1, I(12.4), p. 69]). *Let $M\neq 0$ be a right R-module with the ascending chain condition and the descending* *chain condition on submodules. Suppose*

$$M = M_1 \oplus \cdots \oplus M_r = N_1 \oplus \cdots \oplus N_s$$

are direct decompositions of M *into indecomposable modules* M_i, N_j. *Then* $r = s$, *and, for a suitable ordering of the indices,* $M_i \cong N_i$ $(i = 1, \dots, r)$.

Let $\mathcal{Q}$ be an algebra over a field K. By a right $\mathcal{Q}$-module is meant a K-vector space which is a right $\mathcal{Q}$-module in the ring sense. It is convenient here to call a right $\mathcal{Q}$-module V *irreducible* if $V \neq 0$ and V has no proper $\mathcal{Q}$-submodules (this differs from the normal usage in that a one-dimensional K-subspace U of V such that $U\mathcal{Q} = 0$ is here regarded as irreducible). Suppose $V \neq 0$ is a finite-dimensional right $\mathcal{Q}$-module. Then V is called *completely reducible* if V is the direct sum of irreducible $\mathcal{Q}$-modules. We recall

II [1, V(3.4), p. 467]. *If* $\mathcal{Q}$ *is semisimple, then every right* $\mathcal{Q}$*-module is completely reducible.*

In particular,

III. *If* $\mathcal{Q}$ *is semisimple, then a nonzero indecomposable right* $\mathcal{Q}$*-module is irreducible.*

Suppose now that $\mathcal{Q}$ is a semisimple algebra of $n \times n$ matrices over a field K, and let V be the space of row n-tuples over K. Then V is a right $\mathcal{Q}$-module in the obvious way, so II implies that there exists a natural number s and irreducible $\mathcal{Q}$ subspaces $V_1, \dots, V_s$ of V such that

$$V = V_1 \oplus \cdots \oplus V_s.$$

Choose a basis T_i for each V_i, and let $T = \cup T_i$. Then T is a basis for V. If P is the change of basis matrix (with respect to the standard basis) for T, then for all $A \in \mathcal{Q}$,

$$P^{-1}AP = A_1 \oplus \cdots \oplus A_s$$

where A_i is $n_i \times n_i$ and $n_i = \dim V_i$, $i = 1, \dots, s$.

Suppose further that V is an irreducible $\mathcal{Q}$-module and that $V\mathcal{Q} \neq 0$. Let $\mathcal{K} = \{A \in \mathcal{Q} \mid VA = 0\}$. Then $\mathcal{K}$ is an ideal of $\mathcal{Q}$, and $\mathcal{C} \cong \mathcal{Q}/\mathcal{K}$ is simple. Also,

IV (Wedderburn [1, V(4.4) p. 472]). $\mathcal{C}$ *is isomorphic to the full matrix* *algebra* $M_n(D)$, *where* D *is a division algebra anti-isomorphic to* $Hom_{\mathcal{C}}(V,V)$. *In particular, if* K *is algebraically closed, then* $\mathcal{C} = M_n(K)$, *where* $n = \dim V$.

We note

V. *If* $\mathcal{A}$ *is an algebra of complex* $n \times n$ *matrices with* $\mathcal{A}^* = \mathcal{A}$ (*where* $*$ *denotes complex conjugate transpose*), *then* $\mathcal{A}$ *is semisimple.*

Proof. Since XX^* is nilpotent if and only if $X = 0$, $\mathcal{A}$ has no nonzero nilpotent ideals (cf. the discussion on p. 72 of [2]). ∎

Since a normal matrix A is a real polynomial in A^*, V implies

VI. *If* $\mathcal{S}$ *is a nonempty set of normal* $n \times n$ *matrices, then the algebra generated by* $\mathcal{S}$ *over the real or complex numbers is semisimple.*

We now state our first result:

LEMMA 1. *Let* $\mathcal{S}$ *be a nonempty set of complex* $n \times n$ *matrices, and let* $\mathcal{A}$ *be the* (*complex*) *algebra generated by* $\mathcal{S} \cup \{I\}$. *Suppose that* $\mathcal{A} = \mathcal{A}^*$. *Let* $t, k_1, \ldots, k_t$ *be natural numbers. The following are equivalent*:

(i) *There exists a nonsingular matrix* P *such that, for all* $S \in \mathcal{S}$,

$$P^{-1}SP = M_1(S) \oplus \cdots \oplus M_t(S), \tag{1}$$

where $M_i(S)$ *is* $k_i \times k_i$ $(i = 1, \ldots, t)$.

(ii) *There exists a unitary matrix* U *such that, for all* $S \in \mathcal{S}$,

$$U^{-1}SU = L_1(S) \oplus \cdots \oplus L_t(S), \tag{2}$$

where $L_i(S)$ *is* $k_i \times k_i$ $(i = 1, \ldots, t)$.

Proof. Clearly (ii) implies (i).

Suppose (i) holds. Let V be the space of complex (row) n-tuples. Regard V as a right $\mathcal{A}$-module. By V, $\mathcal{A}$ is semisimple, so by II, V can be written as a direct sum of irreducible $\mathcal{A}$-modules. Also, by (III), every indecomposable $\mathcal{A}$-module is irreducible. Let $V = V_1 \oplus \cdots \oplus V_t$ be the decomposition given by (1). By II, each V_i may be expressed as a direct sum of irreducible $\mathcal{A}$-modules, say

$$V_i = V_{i1} \oplus \cdots \oplus V_{is_i},$$

and the required result will follow once it is established for the decomposition

$$V=\sum_{i=1}^{t}\sum_{j=1}^{s_i}\oplus V_{ij}.$$

Hence we may assume that in (1) the V_i are all irreducible.

If we can show that there exists a decomposition

$$V=W_1\oplus\cdots\oplus W_s, \tag{3}$$

where $W_i\perp W_j$ $(i\neq j)$ and where each W_i is an irreducible $\mathcal{Q}$-module, then I tells us that $s=t$ and that the set $\{\dim W_i|i=1,\dots,t\}$ is the same as the set $\{\dim V_i|i=1,\dots,t\}$, and also that the multiplicity of occurrence of each particular dimension is the same in both decompositions. Let B_i be an orthonormal basis for W_i and let $B=\cup B_i$. Then B is an orthonormal basis for V. Let U be the change of basis matrix (with respect to the standard basis). Then U is unitary and (2) holds. Thus it remains only to establish the existence of the decomposition (3).

Take $W_1=V_1$. Note that since $\mathcal{Q}=\mathcal{Q}^*$, $V_1^\perp$ is $\mathcal{Q}$-invariant, so considering $V_1^\perp$ as an $\mathcal{Q}$-module, we can apply induction to conclude that

$$V_1^\perp=W_2\oplus\cdots\oplus W_s,$$

where W_i is irreducible and $W_i\perp W_j$ $(i\neq j)$.

This completes the proof. ∎

REMARK. We note that the lemma in conjunction with VI shows that a set of normal matrices is simultaneously unitarily quasidiagonalizable if and only if it is simultaneously quasidiagonalizable [i.e., there exists a decomposition of the form (1) with $k_i\leqslant 2$, $i=1,\dots,t$].

Let $x_1,\dots,x_m$ be noncommuting indeterminates. Let S_m denote the symmetric group of degree m regarded as a permutation group on $\{1,\dots,m\}$. The polynomial $(\in\mathbf{Z}[x_1,\dots,x_m])$

$$s_m(x_1,\dots,x_m)=\sum_{\sigma\in S_m}\operatorname{sign}(\sigma)x_{1\sigma}x_{2\sigma}\cdots x_{m\sigma}$$

is called the standard polynomial of degree m. The Amitsur-Levitski theorem [4, (5.1),(5.2), p. 22] states that $s_m(x_1,\dots,x_m)$ is a polynomial identity for

$M_n(C)$ [i.e., $s_m(A_1,\ldots,A_m)=0$ for all choices of $n\times n$ complex matrices $A_1,\ldots,A_m$] if and only if $m\geqslant 2n$. Let $\mathcal{S}$ be a nonempty set of normal matrices. By VI, the complex algebra $\mathcal{A}$ generated by $\mathcal{S}\cup\{I\}$ is semisimple. So, by II, there exists a decomposition of the space V of (row) n-tuples

$$V=V_1\oplus\cdots\oplus V_s$$

such that each V_i is $\mathcal{A}$-irreducible. Let $k_i=\dim V_i$. By IV, $\mathcal{A}$ has an ideal $\mathcal{K}_i$ with $\mathcal{A}/\mathcal{K}_i\cong M_{k_i}(C)$ (here we need the fact that C is algebraically closed), and hence, by the Amitsur-Levitski theorem, if $\mathcal{A}$ satisfies the standard polynomial $s_4(x_1,x_2,x_3,x_4)$, then $k_i\leqslant 2$ for all i, i.e., $\mathcal{S}$ is simultaneously quasidiagonalizable. Conversely, if $\mathcal{S}$ is simultaneously quasidiagonalizable, $\mathcal{A}$ satisfies $s_4(x_1,x_2,x_3,x_4)$. This, in conjunction with the Remark at the end of Lemma 1, gives an affirmative answer to a question of Watters [5, p. 116].

We now prove

THEOREM 1. *Let A,B be $n\times n$ positive semidefinite Hermitian matrices. Then A,B are simultaneously quasidiagonalizable if and only if*

$$[A,B]^2,\quad [A^2,B]^2,\quad [A,B^2]^2\quad \text{commute with}\quad A,\quad B.$$

Proof. The condition is clearly necessary, since the square of a 2×2 commutator is a scalar. Conversely, let $\mathcal{A}$ be the algebra generated by A,B. By Lemma 1, we may assume that $\mathcal{A}$ is simple and noncommutative. Thus $\mathcal{A}=M_n(C)$ and $[A,B]\neq 0$. Since $[A,B]$ is skew-Hermitian, we may also assume that $[A,B]=\mathrm{dg}(\lambda_1,\ldots,\lambda_n)$ with $0=\mathrm{tr}[A,B]=\lambda_1+\cdots+\lambda_n$. By hypothesis $[A,B]^2$ is in the center of $\mathcal{A}$ and is thus scalar; specifically, $\lambda_1^2=\cdots=\lambda_n^2\neq 0$. Therefore n is even and

$$[A,B]=\begin{bmatrix}\lambda I & 0\\ 0 & -\lambda I\end{bmatrix},$$

where I is the identity $(n/2)\times(n/2)$ matrix. Let

$$A=\begin{bmatrix}A_1 & A_2\\ A_2^* & A_3\end{bmatrix},\qquad B=\begin{bmatrix}B_1 & B_2\\ B_2^* & B_3\end{bmatrix}.$$

Since $[A^2,B]=A[A,B]+[A,B]A$ and $[A,B]=\mathrm{dg}(\lambda I,-\lambda I)$, we have $[A^2,B]=2\lambda\,\mathrm{dg}(A_1,-A_3)$. Thus since $[A^2,B]^2$ is scalar, we get $A_1^2=A_3^2$ scalar, and from $[A,B^2]^2$ scalar we get $B_1^2=B_3^2$ scalar. Since A,B are positive semidefinite, we thus find $A_1=A_3=zI$ and $B_1=B_3=\omega I$ for some real numbers z,ω. Now

$$[A,B]=\begin{bmatrix}0 & A_2\\ A_2^* & 0\end{bmatrix}\begin{bmatrix}0 & B_2\\ B_2^* & 0\end{bmatrix}-\begin{bmatrix}0 & B_2\\ B_2^* & 0\end{bmatrix}\begin{bmatrix}0 & A_2\\ A_2^* & 0\end{bmatrix}.$$

$$=\begin{bmatrix}A_2B_2^*-B_2A_2^* & 0\\ 0 & A_2^*B_2-B_2^*A_2\end{bmatrix}.$$

So

$$A_2B_2^*-B_2A_2^*=\lambda I, \qquad A_2^*B_2-B_2^*A_2=-\lambda I. \tag{E}$$

Hence $A_2^*A_2B_2^*=B_2^*A_2A_2^*$, $A_2A_2^*B_2=B_2A_2^*A_2$, and $A_2A_2^*B_2B_2^*=B_2A_2^*A_2B_2^*=B_2B_2^*A_2A_2^*$. Hence $A_2A_2^*,B_2B_2^*$ have a common (nonzero) eigenvector, v say.

Let V be the linear span of

$$\begin{bmatrix}v\\ 0\end{bmatrix},\quad \begin{bmatrix}0\\ A_2^*v\end{bmatrix},\quad \begin{bmatrix}0\\ B_2^*v\end{bmatrix},\quad \begin{bmatrix}B_2A_2^*v\\ 0\end{bmatrix}.$$

Using the equations (E) and the fact that v is an eigenvector of $A_2A_2^*,B_2B_2^*$, we see that V is $\mathcal{Q}$-invariant. Thus $n\leqslant 4$, and so $n=2$ or $n=4$. Suppose $n=4$. Write $A_2=PH$, where P is positive semi-definite and H is unitary. Conjugating A,B by $\begin{bmatrix}H & 0\\ 0 & I\end{bmatrix}$ we may assume that $A_2=P=A_2^*$ without affecting (E). But now $U^{-1}PU$ is diagonal for some unitary U, so conjugating A by $\begin{bmatrix}U^{-1} & 0\\ 0 & U^{-1}\end{bmatrix}$, we may assume that

$$A_2=A_2^*=\begin{bmatrix}a_1 & 0\\ 0 & a_2\end{bmatrix} \qquad \text{where} \quad a_i\geqslant 0.$$

Now B_2 commutes with $A_2A_2^*$, so if $a_1\neq a_2$, then B_2 is also diagonal, B_2^* is diagonal, and A_2,A_2^*,B_2,B_2^* have a common (nonzero) eigenvector, u say. Now the span of $\begin{bmatrix}u\\ 0\end{bmatrix},\begin{bmatrix}0\\ u\end{bmatrix}$ is $\mathcal{Q}$-invariant, forcing $n=2$. Thus we may assume $a_1=a_2$, so A_2 is real scalar. From (E), $B_2-B_2^*$ is scalar (unless $A_2=0$, in which case the result is clear). So B_2,B_2^*,A_2,A_2^* again have a common eigenvector. This completes the proof. ■

THEOREM 2. *Let A be an $n\times n$ positive semidefinite Hermitian matrix, B an $n\times n$ Hermitian matrix. Then A,B are simultaneously quasidiagonalizable if and only if the following matrices commute with A,B:*

$$[A,B]^2,\quad [A^2,B]^2,\quad [A,B^2]^2,\quad [AB,B]^2.$$

Proof. Arguing as in Theorem 1, we may assume that

$$[A,B]=\begin{bmatrix}\lambda I & 0\\ 0 & -\lambda I\end{bmatrix},\qquad A=\begin{bmatrix}A_1 & A_2\\ A_2^* & A_3\end{bmatrix},\qquad B=\begin{bmatrix}B_1 & B_2\\ B_2^* & B_3\end{bmatrix},$$

and as before, $A_1=A_3=zI$ for some real z and $B_1^2=B_3^2$ is scalar. Now

$$[A,B]=\begin{bmatrix}0 & A_2\\ A_2^* & 0\end{bmatrix}\begin{bmatrix}B_1 & B_2\\ B_2^* & B_3\end{bmatrix}-\begin{bmatrix}B_1 & B_2\\ B_2^* & B_3\end{bmatrix}\begin{bmatrix}0 & A_2\\ A_2^* & 0\end{bmatrix}$$

$$=\begin{bmatrix}A_2B_2^*-B_2A_2^* & A_2B_3-B_1A_2\\ A_2^*B_1-B_3A_2^* & A_2^*B_2-B_2^*A_2\end{bmatrix},$$

forcing

$$\begin{aligned}A_2B_2^*-B_2A_2^*&=\lambda I,\\ A_2^*B_2-B_2^*A_2&=-\lambda I,\\ A_2B_3&=B_1A_2.\end{aligned}\tag{X}$$

Also,

$$[AB,B]=[A,B]B=\begin{bmatrix}\lambda & 0\\ 0 & -\lambda\end{bmatrix}\begin{bmatrix}B_1 & B_2\\ B_2^* & B_3\end{bmatrix}=\begin{bmatrix}\lambda B_1 & \lambda B_2\\ -\lambda B_2^* & -\lambda B_3\end{bmatrix}.$$

Hence, from $[AB,B]^2$ scalar, we obtain that $\lambda^2(B_1^2-B_2B_2^*)=\lambda^2(B_3^2-B_2^*B_2)$ is scalar and $\lambda B_1B_2=\lambda B_2B_3$, so, since $\lambda\neq 0$, $B_2B_2^*=B_2^*B_2$ is scalar and $B_1B_2=B_2B_3$. Since we may assume $B_2\neq 0$, this implies B_1 is similar to B_3; this also follows from the fact that $B_1^2=B_3^2$ is scalar and that $0=\operatorname{tr}[A,B^2]=2\lambda\operatorname{tr}(B_1-B_3)$.

Conjugating A,B by $\begin{bmatrix}I & 0\\ 0 & U\end{bmatrix}$ for some suitable unitary U, we may assume that $B_1 = B_3$ without effecting the equations (X). Hence A_2 commutes with B_1 and B_2 commutes with B_1. Let v be a common eigenvector of $A_2A_2^*, B_1$. Then

$$V = \operatorname{span}\left\{\begin{bmatrix} v \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ A_2^* v \end{bmatrix}, \begin{bmatrix} 0 \\ B_2 v \end{bmatrix}, \begin{bmatrix} B_2^* v \\ 0 \end{bmatrix}\right\}$$

is $\mathcal{Q}$-invariant, so $n = 2$ or 4. If $n = 4$ and B_1 is scalar, then the argument can be completed as in Theorem 1. Suppose B_1 is not scalar. We may assume B_1 is diagonal, say

$$B_1 = \sigma\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

for some real σ. Since A_2, B_2 commute with B_1, it follows that A_2, B_2 are both diagonal, so they commute, and the result follows as before. This completes the proof. ■

We note that in Theorem 2, all but one of our conditions are of the form $[A^i, B^j]^2 \in Z(\mathcal{Q})$, where $\mathcal{Q}$ is the algebra generated by A, B. In our next theorem we examine the consequences of conditions of this type.

THEOREM 3. *Let A, B be $n \times n$ Hermitian matrices with A positive semidefinite. The following are equivalent:*

(1) *$[A^i, B]^2$, $[A, B^i]^2$ commute with A, B for $i = 1, 2, 3$,*
(2) *$[A^i, B^j]^2$ commutes with A, B for all $i \geq 1$, $j \geq 1$.*

Furthermore, if (1) *or* (2) *holds there exists $t \geq 1$ and a unitary matrix U such that*

(3) *$U^{-1}AU = C_1 \oplus \cdots \oplus C_t$, $U^{-1}BU = D_1 \oplus \cdots \oplus D_t$, where C_i, D_i are k_i and $k_i \leq 4$ $(1 \leq i \leq t)$.*

A proof of Theorem 3 can be obtained using the ideas of the proofs of Theorems 1, 2 and is therefore omitted.

We now give an example of a pair of matrices satisfying the hypotheses of Theorem 3.

EXAMPLE. Let

$$A = \begin{bmatrix} 2I & I \\ I & 2I \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & B_2 \\ B_2^* & B_1 \end{bmatrix}$$

where

$$B_1=\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \qquad B_2=\begin{bmatrix} 1+i & 0 \\ 0 & -1+i \end{bmatrix}.$$

Then $[A^i,B^j]^2$ is scalar for all $i\geqslant 1$, $j\geqslant 1$, and A,B generate $M_4(C)$.

We now come to our main result on quasidiagonalizability.

THEOREM 4. *Let A,B be $n\times n$ Hermitian matrices, and let $\mathcal{A}$ be the algebra generated by A,B. Then A,B are simultaneously quasidiagonalizable if (and only if) the squares of the following matrices belong to the center of $\mathcal{A}$:*

$$[A^2,B],\quad [A^4,B],\quad [A^8,B],\quad [A^2B,B],\quad [A^2,AB], [A^2,B^2],$$

$$[A,B^2],\quad [A,B^4],\quad [A,B^8],\quad [A,B^2A],\quad [B^2,AB].$$

REMARK. It seems likely that some of the above commutators are redundant, and it would be interesting to determine a minimal set.

Proof. We may assume that $\mathcal{A}=M_n(C)$ and that $n>1$. We may also assume that $[A^2,B]$ is diagonal and thus that

$$[A^2,B]=\begin{bmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{bmatrix} \qquad \left(I \text{ the } \frac{n}{2}\times\frac{n}{2} \text{ identity}\right).$$

If $\lambda=0$, then A^2 commutes with B and thus $A^2\in Z(\mathcal{A})$, and therefore A^2 is scalar; this fact will be used later.

Suppose $\lambda\neq 0$. Let

$$A^2=\begin{bmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{bmatrix}.$$

Now

$$[A^4,B]=A^2[A^2,B]+[A^2,B]A^2=\begin{bmatrix} 2\lambda A_1 & 0 \\ 0 & -2\lambda A_3 \end{bmatrix},$$

so since $[A^4,B]^2$ is scalar, $A_1^2=A_3^2$ is scalar.

Thus $A_1 = A_3 = zI$ for some real z, since A^2 is positive semidefinite. Now

$$A^4 = \begin{pmatrix} z^2I + A_2A_2^* & 2zA_2 \\ 2zA_2^* & z^2I + A_2^*A_2 \end{pmatrix},$$

so

$$[A^8, B] = \begin{bmatrix} 2z(z^2I + A_2A_2^*) & 0 \\ 0 & -2z(z^2I + A_2^*A_2) \end{bmatrix}.$$

Since $[A^8, B]^2$ is scalar,

$$(z^2I + A_2A_2^*)^2 = (z^2I + A_2^*A_2)^2$$

is scalar, and, since $z^2I + A_2A_2^*$ is positive semidefinite,

$$z^2I + A_2A_2^* = z^2I + A_2^*A_2$$

is scalar. Hence $A_2A_2^* = A_2^*A_2$ is scalar. So $A_2 = \omega U$ for some real ω and unitary matrix U. Conjugating A, B by $\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}$, we may assume that

$$A^2 = \begin{bmatrix} zI & \omega I \\ \omega I & zI \end{bmatrix}.$$

Let

$$B = \begin{bmatrix} B_1 & B_2 \\ B_2^* & B_3 \end{bmatrix}.$$

Since

$$[A^2, B] = \begin{bmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{bmatrix},$$

we now have $B_2^* - B_2 = \lambda I$ and $B_3 = B_1$.

Now

$$[A^2, B^2] = \begin{bmatrix} 2\lambda B_1 & 0 \\ 0 & -2\lambda B_1 \end{bmatrix},$$

so, since $[A^2, B^2]^2$ is scalar, B_1^2 is scalar.

Also $[A^2B,B]^2$ is scalar, so $B_1^2-B_2B_2^*=B_1^2-B_2^*B_2$ is scalar and $B_1B_2=B_2B_1$. Since B_1^2 is scalar, $B_2B_2^*=B_2^*B_2$ is scalar. Now B_1,B_2 have a common eigenvector, say x, and this is also an eigenvector of B_2^*. Thus

$$W=\operatorname{span}\left\{\begin{bmatrix} x \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ x \end{bmatrix}\right\}$$

is a $\mathcal{B}$-invariant subspace of C^n, where $\mathcal{B}$ is the algebra generated by A^2,B. So $\mathcal{B}$ is a direct sum of copies of C and $M_2(C)$. This is also the case if $\lambda=0$ (of course, only copies of C arise then).

Let $\mathcal{C}$ be the algebra generated by A,B^2. Reversing the roles of A,B (note that the hypotheses of the theorem are symmetric in A,B), we see that there is a $\mathcal{C}$-invariant subspace V of C^n with $1\leqslant \dim V\leqslant 2$. If A^2 commutes with B and B^2 commute with A, then A^2 is scalar, B^2 is scalar, and $\mathcal{C}$ is spanned as a vector space by A,B,AB,BA, and the conclusion of the theorem holds. Thus we may assume that at least one of $[A^2,B],[A,B^2]$ is nonzero, and thus we may assume that $\lambda\neq 0$ (otherwise replace A by B to start with). Let

$$0\neq\begin{bmatrix} h \\ k \end{bmatrix}\in V.$$

Since V is A^2-invariant, $\begin{bmatrix} k \\ h \end{bmatrix}\in V$, and thus either (i) V consists of elements of the form $\begin{bmatrix} h \\ h \end{bmatrix}$, or (ii) V is spanned by two elements of the form $\begin{bmatrix} h_1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ h_2 \end{bmatrix}$. Suppose (i) holds. From $B^2\begin{bmatrix} h \\ h \end{bmatrix}\in V$, we get

$$\begin{bmatrix} B_1^2+B_2B_2^* & 2B_1B_2 \\ 2B_1B_2^* & B_1^2+B_2B_2^* \end{bmatrix}\begin{bmatrix} h \\ h \end{bmatrix}\in V$$

and thus, since $B_2B_2^*$ is scalar,

$$\begin{bmatrix} B_1B_2h \\ B_1B_2h \end{bmatrix}\in V.$$

If (ii) holds, taking $B^2\begin{bmatrix} 0 \\ h \end{bmatrix}$ gives

$$\begin{bmatrix} B_1B_2h \\ 0 \end{bmatrix}\in V.$$

Suppose on the other hand, no nonzero element of the form $\begin{bmatrix} h \\ h \end{bmatrix} \in V$. Then V consists of elements of the form $\begin{bmatrix} h \\ -h \end{bmatrix}$. Applying B^2, we get

$$\begin{bmatrix} (B_1^2 + B_2 B_2^*)h - 2B_1 B_2 h \\ 2B_1 B_2^* h - (B_1^2 + B_2 B_2^*)h \end{bmatrix} \in V,$$

implying that

$$\begin{bmatrix} B_1 B_2 h \\ -B_1 B_2^* h \end{bmatrix} \in V.$$

But $B_2^* - B_2 = \lambda I$ is scalar and thus $\begin{bmatrix} B_1 B_2 h \\ -B_1 B_2^* h \end{bmatrix}$ is not of the form $\begin{bmatrix} u \\ -u \end{bmatrix}$ unless $B_1 h = 0$. But B_1^2 is scalar and nonzero, so $B_1 h = 0$ implies $h = 0$. This eliminates the possibility that V has no elements of the form $0 \neq \begin{bmatrix} h \\ h \end{bmatrix}$. Hence there exists an element $\begin{bmatrix} v \\ v \end{bmatrix} \in V$ such that v is an eigenvector of $B_1 B_2$.

We now claim that any eigenvector of $B_1 B_2$ is an eigenvector of B_1 and an eigenvector of B_2. Recall that B_1^2 is scalar, that $B_2 B_2^*$ is scalar, that $B_2 - B_2^*$ is scalar $(-\lambda I)$, and that $B_1 B_2 = B_2 B_1$. Diagonalizing B_1, B_2 simultaneously by a unitary matrix, we may assume

$$B_1 = \begin{bmatrix} \sigma I_1 & 0 \\ 0 & -\sigma I_2 \end{bmatrix}, \qquad I_1 \, (s \times s),$$

with $\sigma \neq 0$ (if $\sigma = 0$, the result follows from argument of Theorem 2), and

$$B_2 = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{bmatrix}.$$

Let

$$\omega = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

be an eigenvector of $B_1 B_2$ (α being $s \times 1$). Say $\alpha \neq 0$. Then $B_2 \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$ is a multiple of $\begin{bmatrix} \alpha \\ 0 \end{bmatrix}$, and thus, without loss of generality,

$$B_2\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \lambda_1\begin{bmatrix} \alpha \\ 0 \end{bmatrix}.$$

But then

$$B_2\begin{bmatrix} 0 \\ \beta \end{bmatrix} = -\lambda_1\begin{bmatrix} 0 \\ \beta \end{bmatrix},$$

so if $\beta \neq 0$, then $\lambda_1, -\lambda_1$ are eigenvalues of B_2. But since $B_2^* - B_2 = \lambda I$ and $B_2B_2^*$ is scalar, say tI, the minimal polynomial of B_2 has degree at most 2, so the only eigenvalues of B_2 are $\pm\lambda_1$. Now if $t=0$, then $B_2=0$ and $\lambda=0$—a contradiction. So $t\neq 0$ and $B_2^* = tB_2^{-1}$. So from $B_2^* - B_2 = \lambda I$ we get $B_2^2 + \lambda B_2 - tI = 0$, so $\lambda = 0$. This contradiction forces $\beta = 0$. A similar argument shows that if $\beta \neq 0$, then $\alpha = 0$. This establishes that ω is an eigenvector of both B_1 and B_2.

If we choose

$$0 \neq \begin{bmatrix} v \\ v \end{bmatrix} \in V$$

such that v is an eigenvector of B_1, B_2, then

$$\operatorname{span}\left\{\begin{bmatrix} v \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v \end{bmatrix}\right\}$$

is an $\mathcal{Q}$-invariant subspace. This completes the proof. ∎

In conclusion, we remark that in [3] we obtain an additive criterion for the real algebra generated by a pair of normal matrices to be a direct sum of copies of R, C and the real quaternions, and hence give another criterion for a pair of normal matrices to be simultaneously quasidiagonalizable.

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